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For foreign students

МЕТОДИЧНІ ВКАЗІВКИ

Одеса - 2026

Методичні вказівки розроблено кандидатом фізико-математичних наук **Кусік Людмилою Ігорівною** – доцентом кафедри вищої математики Одеського національного морського університету. Конспект містить основний теоретичний матеріал з теорії рядів. Наведено приклади з розв'язаннями.

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Series

§1. Properties of a numerical series

Definition. Consider a numbered set of numbers $a_1, a_2, \ldots, a_n, \ldots$. Let us call it a numerical sequence. The notation $a_1, a_2, \ldots, a_n, \ldots = \{a_n\}_{n=1}^{\infty}$ is used. It is advisable to define a numerical sequence by the formula of its common member, that is $a_n = f(n)$, by the function depending on the natural number n. For example, the formula for the general term $a_n = \frac{1}{n^2}$ gives the sequence $1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{n^2}, \ldots = \left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$. A numerical sequence is called increasing if $a_n < a_{n+1}$ and decreasing if $a_n > a_{n+1}$ $\left(n = \overline{1;\infty}\right)$. If the sequence increases or decreases, then it is called monotonic. A numerical sequence $\{a_n\}_{n=1}^{\infty}$ is said to be bounded if there exists a number $M\,(M > 0)$ such that $|a_n| \le M$ $\left(n = \overline{1;\infty}\right)$. The number A is called the limit of a numerical sequence $\{a_n\}_{n=1}^{\infty}$ if for any positive number ε we can choose a number N, such that $|a_n - A| < \varepsilon$ $(n \ge N)$. If a sequence has a limit, then it is said to converge.

Theorem 1 (Weierstrass) A monotone and bounded numerical sequence has a limit.

Definition. A numerical series is the sum of the terms of a numerical sequence. It is written

$$a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n . \quad (1)$$

Numbers $a_1, a_2, \ldots, a_n, \ldots$ are called terms of a series, a_n is called its general term.

The sum of the first n terms of the series is called its n-partial sum. We introduce the notation $S_n = a_1 + a_2 + \ldots + a_n$. By assigning n values $1, 2, 3, \ldots$, we obtain a numerical sequence $S_1, S_2, \ldots, S_n, \ldots = \{S_n\}_{n=1}^{\infty}$ that is a sequence of partial sums of the series (1). A series is said to converge if the sequence of its partial sums converges, that is, if there is a finite limit $\lim\limits_{n\to\infty} S_n = S$. A number S is called the sum of a numerical series (1), that is, it is considered that

$$a_1 + a_2 + \ldots + a_n + \ldots = S .$$

Examples: study the convergence of a series

1) $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Solving. Let's write down the n-th partial sum $S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n\cdot(n+1)}$.

Note that

$$a_1 = \frac{1}{1\cdot 2} = 1 - \frac{1}{2},\ \ a_2 = \frac{1}{2\cdot 3} = \frac{1}{2} - \frac{1}{3}, \ldots, a_n = \frac{1}{n} - \frac{1}{n+1}.$$

Then

$$S_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \ldots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1};\ \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1.$$

Therefore, the series converges and its sum is 1.

2) $\sum_{n=1}^{\infty} b_1 q^{n-1}$, b_1, q are constants.

Solving. Such a series is called geometric, because it is the sum of the terms of a geometric progression $b_1, b_1 q, b_1 q^2, \ldots, b_1 q^{n-1}, \ldots$. Let's use the well-known formula for the sum of the first n terms of a geometric progression: $S_n = b_1 \cdot \frac{1-q^n}{1-q}$.

Let $|q| < 1$, then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} b_1 \cdot \frac{1-q^n}{1-q} = \frac{b_1}{1-q}$. Therefore, a geometric series converges if $|q| < 1$, and its sum is equal to $\frac{b_1}{1-q}$.

If $q > 1$ then $\lim_{n\to\infty} q^n = \infty$ and $\lim_{n\to\infty} S_n = \infty$. If $q < -1$ then $\lim_{n\to\infty} S_n$ does not exist because it does not exist $\lim_{n\to\infty} q^n$. Thus, the geometric series diverges at $|q| > 1$. It remains to consider only the cases $q = \pm 1$. We will show that in these cases the series diverges. If $q = 1$, then the series has the form $b_1 + b_1 + \ldots + b_1 + \ldots$. Then $S_n = b_1 \cdot n$, $\lim_{n\to\infty} S_n = \infty$, therefore the series diverges. If $q = -1$, then the series $b_1 - b_1 + b_1 - b_1 + \ldots + (-1)^{n-1} b_1 + \ldots$ also diverges, since the sequence of its partial sums has the form $0, b_1, 0, b_1, \ldots$, i.e., has no limit.

Basic properties of series

Theorem 2. If all terms of a convergent series with the sum S are multiplied by the any number k, then we obtain a convergent series whose sum is equal to $k \cdot S$.

***Proof*.** Let the series $\sum_{n=1}^{\infty} a_n$ converges, S is its sum, S_n is its n- th partial sum. Let us find the limit of the n- th partial sum of the series, which we obtain after multiplying each term by k:

$$\lim_{n\to\infty}(ka_1 + ka_2 + \ldots + ka_n) = \lim_{n\to\infty} k(a_1 + a_2 + \ldots + a_n) = k \lim_{n\to\infty} S_n = k \cdot S.$$

The theorem is proved.

Theorem 3. If the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge and their sums are respectively equal to S and σ, then the series obtained by their termwise addition and subtraction converge. In this case $\sum_{n=1}^{\infty}\left(a_n \pm b_n\right) = S \pm \sigma$.

Proof. $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\left(a_i \pm b_i\right) = \lim\limits_{n\to\infty}\sum_{i=1}^{n} a_i \pm \lim\limits_{n\to\infty}\sum_{i=1}^{n} b_i = S \pm \sigma$, which had to be proven.

Definition. Let we discard the first n terms in the numerical series (1). The series that remains $a_{n+1} + a_{n+2} + \ldots + a_{n+k} + \ldots$ we call the n-th remainder of the series (1).

Theorem 4. The series (1) and its remainder both converge or both diverge.

Proof. Let's fix the number n. Let $S_n = a_1 + a_2 + \ldots + a_n$ is the n-th the partial sum of the series (1), $\sigma_k = a_{n+1} + a_{n+2} + \ldots + a_{n+k}$ is the partial sum of its remainder. Then $S_{n+k} = S_n + \sigma_k$. If the series (1) converges and its sum be equal to S, then there exists a finite limit $\lim\limits_{k\to\infty}\sigma_k = \lim\limits_{k\to\infty}\left(S_{n+k} - S_n\right) = S - S_n$. That is, the n-th remainder also converges. Now suppose that the n-th remainder of the series (1) converges. Let us denote its sum by r_n. Then $\lim\limits_{k\to\infty} S_{n+k} = S_n + \lim\limits_{k\to\infty}\sigma_k = S_n + r_n$ is finite number. The last equality means that the series (1) converges, its sum is denoted by S . Therefore $S = S_n + r_n$. Now let the series (1) diverges. Then its n-th remainder also diverges. After all, in the case of convergence of the remainder, the series (1) would also converge. By analogy, we obtain: from the divergence of the remainder, the divergence of the series follows. The theorem is proved.

Consequences. 1. Theorem 4 can also be understood in this way: discarding several of the first terms of a series or adding a finite number of new terms to the beginning of a series does not affect its convergence, but can only change its sum.

2. A numerical series converges if and only if the sum r_n of its n-th remainders tends to zero at $n \to \infty$. After all, from equality $S = S_n + r_n$ it follows $r_n = S - S_n$ or $\lim\limits_{n\to\infty} r_n = S - \lim\limits_{n\to\infty} S_n = S - S = 0$.

Theorem 5 (necessary convergence condition of a series). If the series $\sum_{n=1}^{\infty} a_n$ converges, then its common term tends to 0 at $n \to \infty$.

Proof. Let be S the sum of the series. Since $a_n = S_n - S_{n-1}$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$$

Remark. It is impossible to prove the convergence of series using this theorem, we can only state that the series diverges if $\lim\limits_{n\to\infty} a_n \neq 0$. After all, for any convergent series $\lim\limits_{n\to\infty} a_n = 0$.

Problem. Study the convergence of the series $\sum\limits_{n=1}^{\infty} \frac{4n-3}{n+1}$.

Solving. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{4n-3}{n+1} = 4 \neq 0$. The series diverges.

Theorem 6. If a series $\sum\limits_{n=1}^{\infty} a_n$ converges, then it is possible (without permuting its terms) to combine them into groups with a finite number of terms using parentheses. This operation does not violate the convergence of the series and does not change its sum.

§2. Series with positive terms

Consider a series whose terms are all positive

$$\sum_{n=1}^{\infty} a_n, \;\; a_n > 0 \;\; \left(n = \overline{1,\infty}\right). \tag{2}$$

Theorem 7 (necessary and sufficient convergence condition of a series with positive terms). In order for the series (2) to converge, it is necessary and sufficient that the sequence of its partial sums $\{S_n\}_{n=1}^{\infty}$ be bounded from above, i.e. there exists a number $M\,(M > 0)$ such that $S_n \leq M$, $n = \overline{1,\infty}$.

Proof. *Necessity.* Let the series (2) converges. Then the sequence of its partial sums has a finite limit. This sequence increases monotonically, since $a_n > 0$ $\left(n = \overline{1,\infty}\right)$. Then it must be bounded from above, since otherwise $\lim\limits_{n\to\infty} S_n = \infty$.

Sufficiency. If the sequence $\{S_n\}_{n=1}^{\infty}$ is bounded above, then, given that it monotonically increases, we obtain: this sequence has a finite limit by the Weierstrass theorem. Therefore, the series converges.

The theorem is proved.

Theorem 8 (integral convergence test). Let at $x \in [1;\infty)$ the terms of the series (2) coincide with the value of a continuous, nonincreasing, nonnegative function $f(x)$, if $x = n$, i.e. $a_n = f(n)$, $n = \overline{1,\infty}$. Then the series (2) converges or diverges simultaneously with the improper integral $\int\limits_1^{\infty} f(x)\,dx$.

Proof. Let n be some natural number. Consider a curved trapezoid bounded by lines $x=1,\ x=n,\ y=0,\ y=f(x)$. Its area is equal to the definite integral $\int_1^n f(x)dx$.

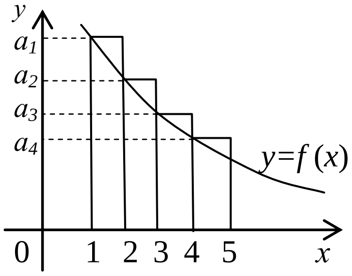

Fig 1

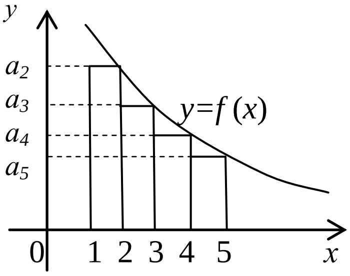

Fig 2

Let us enclose this curvilinear trapezoid in a stepped figure (see Figure 1 for $n=5$), consisting of rectangles, one of whose sides is equal to 1 and the other $a_1, a_2, \ldots, a_{n-1}$. Then the area of this figure is equal to $\sigma_1 = a_1 + a_2 + \ldots + a_{n-1}$. Also consider the stepped figure inscribed in a curved trapezoid, shown in Figure 2. Its area is $\sigma_2 = a_2 + a_3 + \ldots + a_n$. Since $\sigma_2 < \int_1^n f(x)dx < \sigma_1$ then $S_n - a_1 < \int_1^n f(x)dx < S_n - a_n$, where S_n is the n-th partial sum of the series (2). From the last inequality we have

$$S_n < \int_1^n f(x)dx + a_1; \tag{3}$$

$$S_n > \int_1^n f(x)dx + a_n. \tag{4}$$

Let the improper integral $\int_1^\infty f(x)dx$ converges, then the infinite curvilinear trapezoid constructed on the interval $[1;\infty)$ has finite area S. Therefore $\lim_{n\to\infty} \int_1^n f(x)dx = \int_1^\infty f(x)dx = S$, $\int_1^n f(x)dx \le S$. Then from inequality (3) we have $S_n < S + a_1$, that is, the partial sums of the series (2) are bounded from above. Thus, the series converges. If the improper integral $\int_1^\infty f(x)dx$ diverges , that is,

$\lim_{n\to\infty}\int_1^n f(x)dx = +\infty$, then from inequality (4) it follows that is $\{S_n\}_{n=1}^{\infty}$ unbounded. So the series diverges. Now let the series (2) converges, then the improper integral also converges. After all, in the opposite case, the series would also diverge. By analogy, the fact is proved that the divergence of the integral follows from the divergence of the series.

Example. Study the convergence of a series $\sum_{n=1}^{\infty}\frac{1}{n^p}$, where p is constant.

Solving. This series is called a generalized harmonic series, and in the case $p=1$ is called a harmonic series.

According to the integral sign, the question of its convergence is solved using the improper integral $\int_1^{\infty}\frac{dx}{x^p}$. Since this integral converges at $p>1$ and diverges at $p\le 1$, then the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges in the case $p>1$ and diverges at $p\le 1$.

Theorem 9 (first comparison test). Let the series $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ be satisfied with the inequality $0<a_n\le b_n$ $\left(n=\overline{1,\infty}\right)$. Then the convergence of the series $\sum_{n=1}^{\infty}b_n$ implies the convergence of the series $\sum_{n=1}^{\infty}a_n$. If the series $\sum_{n=1}^{\infty}a_n$ diverges, then the series $\sum_{n=1}^{\infty}b_n$ also diverges.

Proof. Let us introduce the notation $S_n = a_1 + a_2 + \ldots + a_n$, $\sigma_n = b_1 + b_2 + \ldots + b_n$. Obviously, $S_n \le \sigma_n$. If the series $\sum_{n=1}^{\infty}b_n$ converges, then the sequence $\{\sigma_n\}_{n=1}^{\infty}$ is bounded above, then the sequence $\{S_n\}_{n=1}^{\infty}$ is also bounded above. Therefore, the series $\sum_{n=1}^{\infty}a_n$ also converges. Let the series $\sum_{n=1}^{\infty}a_n$ diverges, then the series $\sum_{n=1}^{\infty}b_n$ cannot converge. After all, in this case (according to the proof above) the series $\sum_{n=1}^{\infty}a_n$ would also converge.

Examples. Study the convergence of the series:

1) $\sum_{n=1}^{\infty}\frac{1}{n\cdot 2^n}$.

Solving. Since $\frac{1}{n\cdot 2^n}\le\frac{1}{2^n}$ $\left(n=\overline{1,\infty}\right)$ and the geometric series $\sum_{n=1}^{\infty}\frac{1}{2^n}$ $\left(q=\frac{1}{3}<1\right)$ converges, then the series $\sum_{n=1}^{\infty}\frac{1}{n\cdot 2^n}$ also converges.

2) $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n(n-1)}}$.

Solving. It is obvious that $\frac{1}{\sqrt{n(n-1)}}\ge\frac{1}{n}$ $\left(n=\overline{2,\infty}\right)$. The harmonic series $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges, therefore the series $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n(n-1)}}$ also diverges.

Remark. Theorem 9 is also valid in the case when the inequality $0<a_n\le b_n$ is not satisfied for all $n=\overline{1,\infty}$, but starting from some number k, i.e., for $n=\overline{k,\infty}$. After all, discarding several first terms of the series does not affect its convergence.

Theorem 10 (second comparison test). If the series with positive terms $\sum_{n=1}^{\infty}a_n\left(a_n>0\right)$ and $\sum_{n=1}^{\infty}b_n\left(b_n>0\right)$ are such that there exists a finite $\lim_{n\to\infty}\frac{a_n}{b_n}=l$ $\left(l\ne 0,\, l\ne\infty\right)$, then both series $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ converge or both diverge.

Proof. The fact that $\lim_{n\to\infty}\frac{a_n}{b_n}=l$ means that for any number ε $\left(\varepsilon>0\right)$ there is a number N such that for all $n>N$ the inequality $\left|\frac{a_n}{b_n}-l\right|<\varepsilon$ holds. Then $-\varepsilon<\frac{a_n}{b_n}-l<\varepsilon$; $-\varepsilon+l<\frac{a_n}{b_n}<\varepsilon+l$; $\left(l-\varepsilon\right)b_n<a_n<\left(l+\varepsilon\right)b_n$. If the series $\sum_{n=1}^{\infty}b_n$ converges, then by Theorem 2 the series $\sum_{n=1}^{\infty}\left(l+\varepsilon\right)\,b_n$ converges. Then by Theorem 9 the series $\sum_{n=1}^{\infty}a_n$ converges. Let the series $\sum_{n=1}^{\infty}a_n$ converge, then from the inequality $b_n < \frac{1}{1-\varepsilon} a_n$ (ε is considered so small that $0 < \varepsilon < l$) it follows that the series $\sum_{n=1}^{\infty} b_n$ converges.

Let any of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge, then according to the above, the second series cannot converge, because in this case the first one would also converge.

Example. Study the convergence of the series $\sum_{n=1}^{\infty} \sin\frac{\pi}{n^2+1}$.

Solving. Consider the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (see generalized harmonic series, $p=2$). Calculate the limit $\lim_{n\to\infty}\left(\sin\frac{\pi}{n^2+1} : \frac{1}{n^2}\right)$, putting the infinitely small value $\sin\frac{\pi}{n^2+1}$ by its equivalent $\frac{\pi}{n^2+1}$. Then $\lim_{n\to\infty}\left(\sin\frac{\pi}{n^2+1} : \frac{1}{n^2}\right) = \lim_{n\to\infty}\frac{\pi n^2}{n^2+1} = \pi$. By Theorem 10, the given series behaves in the same way as the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$, i.e., it converges.

Theorem 11 (d'Alembert or ratio test). Let the series with positive terms (2) such that there exists (finite or infinite limit) $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = D$ $(D \ge 0)$. Then for $D<1$ the series converges, for $D>1$ it diverges.

Proof. From the definition of the limit we have: for any number $\varepsilon(\varepsilon > 0)$ there is a number N such that $\left|\frac{a_{n+1}}{a_n} - D\right| < \varepsilon$ for $n \ge N$. Then

$$(D-\varepsilon)a_n < a_{n+1} < (D+\varepsilon)a_n. \tag{5}$$

Let us denote by $q = D + \varepsilon$. Let $D<1$, ε be considered so small that $0<q<1$. Then

$a_{n+1} < q\cdot a_n$ $(n \ge N)$, i.e.,

$a_{N+1} < q\cdot a_N$;

$a_{N+2} < q\cdot a_{N+1} < q^2\cdot a_N$;

$a_{N+3} < q\cdot a_{N+2} < q^3\cdot a_N \dots$.

This means that the terms of the series $\sum_{k=1}^{\infty} a_{N+k}$ (the remainder of the series $\sum_{n=1}^{\infty} a_n$) are less than the corresponding terms of the geometric convergent series

$\sum_{k=1}^{\infty} a_N q^k \ (0<q<1)$. By the comparison theorem, the series $\sum_{k=1}^{\infty} a_{N+k}$ also converges. Then the series $\sum_{n=1}^{\infty} a_n$ also converges.

Let $D>1$, ε let us take so small that $D-\varepsilon>1$. Then from inequality (5) we have $a_{n+1}>(D-\varepsilon)a_n>a_n \quad (n\geq N)$. Thus, the terms of the series $\sum_{n=1}^{\infty} a_n$ after the N-th term increase with increasing number n. This means that $\lim_{n\to\infty} a_n=+\infty\neq 0$. Therefore, the series diverges.

Remark. If D=1, then d'Alembert test doesn't answer the question of series convergence. In this case, another test must be used.

Example. Study the convergence of the series $\sum_{n=1}^{\infty} \frac{5^n}{n!}$.

Solving. The common term of the series $a_n=\frac{5^n}{n!}$, then the next term $a_{n+1}=\frac{5^{n+1}}{(n+1)!}=\frac{5^n\cdot 5}{n!\,(n+1)}$. Then $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{5^n\cdot 5}{n!\,(n+1)}\cdot\frac{n!}{5^n}=\lim_{n\to\infty}\frac{5}{n+1}=0<1.$ So the series converges.

Theorem 12 (Cauchy root test). Let the series with positive terms $\sum_{n=1}^{\infty} a_n \ (a_n>0)$ be such that there exists a limit $\lim_{n\to\infty}\sqrt[n]{a_n}=K$ (finite or infinite). Then for $K<1$ the series converges, and for $K>1$ it diverges. For K=1 another test must be used.

We propose to prove this theorem yourself by analogy with the proof of the d'Alembert test.

Theorem 13. If the series $\sum_{n=1}^{\infty} a_n \quad (a_n>0)$ converges, and its sum is S, then the series obtained by any permutation of its terms has the same sum.

Proof. Let S_n be the n-th partial sum of the series $\sum_{n=1}^{\infty} a_n$. Let us rearrange its terms in some way and find the sum of the first m terms of the obtained series: $\sigma_m=a_{n_1}+a_{n_2}+\ldots+a_{n_m}$. Let us denote by l the largest of the numbers $n_1, n_2,\ldots,n_m$. Then $\sigma_m\leq S_l\leq S$. That is, the sequence of partial sums of the obtained series $\{\sigma_m\}_{m=1}^{\infty}$ is bounded from above. Therefore, the series converges, and its sum $\sigma=\lim_{m\to\infty}\sigma_m$ satisfies the inequality $\sigma\leq S$. Thus, the sum σ of the

series obtained by rearranging the terms of the series $\sum_{n=1}^{\infty} a_n \ (a_n > 0)$ cannot be greater than S. But it cannot be less either S. After the inverse permutation we will get the series $\sum_{n=1}^{\infty} a_n$ the sum of which S must satisfy the inequality $S \le \sigma$. Thus $S = \sigma$.
The theorem is proved.

Self-study task
Study the convergence of the series:
1) $\sum_{n=1}^{\infty}\left(1-\frac{3}{n}\right)^n$; 2) $\sum_{n=3}^{\infty}\frac{1}{n\ln^5 n}$; 3) $\sum_{n=1}^{\infty}\frac{2n-1}{\sqrt{5^n+3}}$; 4) $\sum_{n=1}^{\infty}\frac{\sqrt[3]{27n+4}}{5n^2-n}$; 5) $\sum_{n=1}^{\infty}\left(\frac{2n+5}{3n-1}\right)^n$;
6) $\sum_{n=1}^{\infty}\frac{\cos^4 n}{n^3+2n}$; 7) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}\ 2^{\sqrt{n}}}$; 8) $\sum_{n=2}^{\infty}\frac{4}{\sqrt[5]{\ln n}}$; 9) $\sum_{n=1}^{\infty}\frac{n!}{(2n+6)\ 4^n}$; 10) $\sum_{n=1}^{\infty}\frac{\sqrt{4n+3}}{(2n)!}$.

§3. Series with arbitrary terms

If all terms of the numerical series $\sum_{n=1}^{\infty} a_n$ are negative, then (by Theorem 2) the question of its convergence is solved using a series with positive terms $\sum_{n=1}^{\infty}(-a_n)$, because $\sum_{n=1}^{\infty} a_n = -\sum_{n=1}^{\infty}(-a_n)$. Let the members of the series $\sum_{n=1}^{\infty} a_n$ be both positive and negative. If there are a finite number of negative terms, then we discard as many first terms of the series as there are only positive terms in the remainder. We study this remainder for convergence. A series with a finite number of positive terms is investigated in a similar way. Therefore, when studying series with arbitrary terms, it makes sense to consider only those series that have an infinite number of both positive and negative terms. It is such series that will be considered in the following.

Theorem 14. Let the numerical series $\sum_{n=1}^{\infty} a_n$ be such that the series composed of the moduli of its terms $\sum_{n=1}^{\infty}|a_n|$ is convergent. Then the series $\sum_{n=1}^{\infty} a_n$ converges as well.

Proof. Let S_n is the n-th partial sum of the series $\sum_{n=1}^{\infty} a_n$ and σ_n is the n-th partial sum of the series $\sum_{n=1}^{\infty}|a_n|$. Let S_n' denote the sum of the positive terms among the first n terms of the series $\sum_{n=1}^{\infty} a_n$ and let S_n'' denote the sum of the moduli of the negative terms. Then $S_n = S_n' - S_n''$, $\sigma_n = S_n' + S_n''$. Since the series $\sum_{n=1}^{\infty} |a_n|$ has a sum (denote it by σ), then $S_n' \le \sigma_n \le \sigma$, $S_n'' \le \sigma_n \le \sigma$. Thus, the sequences $\{S_n'\}_{n=1}^{\infty}$ and $\{S_n''\}_{n=1}^{\infty}$ are monotone and bounded. Therefore, each of them has a limit. Let us denote these limits by S' and S'' respectively. Then there exists a limit S of the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$: $S = \lim_{n\to\infty} S_n = S' - S''$. That is, this series converges.

Remark. A series can converge even if the series composed of the modules of its items diverges.

Definition. If the series composed of the moduli of the members of the series $\sum_{n=1}^{\infty} a_n$ converges, the series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent. If the series $\sum_{n=1}^{\infty} a_n$ converges, and the series $\sum_{n=1}^{\infty} |a_n|$ diverges, then the series $\sum_{n=1}^{\infty} a_n$ is called conditionally convergent.

§4. Series with alternating signs

Let us consider a series whose members are alternately positive and negative. That is, from a numerical sequence of positive numbers $\{a_n\}_{n=1}^{\infty}$ $(a_n > 0)$ we construct a series of the form

$$a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n-1} a_n + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

or (6)

$$-a_1 + a_2 - a_3 + \ldots + (-1)^n a_n + \ldots = \sum_{n=1}^{\infty} (-1)^n a_n .$$

Let's call it a series with alternating signs.

Theorem 15 (Leibniz test). If the series (6) is such that

1) $\lim_{n\to\infty} a_n = 0$;

2) the moduli of its terms decrease as the number increases, i.e., $a_1 > a_2 > \ldots > a_n > a_{n+1} > \ldots$,

then the series converges.

Proof. Consider a series with alternating signs $a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n-1} a_n + \ldots,\ a_n > 0$. Write its partial sum with an even number: $n = 2k\ (k \in N)$: $S_{2k} = a_1 - a_2 + a_3 - a_4 + \ldots + a_{2k-1} - a_{2k}$. From the second condition of the theorem, it follows that $a_1 - a_2 > 0,\ a_3 - a_4 > 0, \ldots, a_{2k-1} - a_{2k} > 0$. Then $S_{2k} > 0$ and this partial sum increases with increasing k. On the other hand $S_{2k} = a_1 - \big[(a_2 - a_3) + {}$ $+ (a_4 - a_5) + \ldots + (a_{2k-2} - a_{2k-1}) + a_{2k}\big] < a_1$. Therefore, the sequence $\{S_{2k}\}_{k=1}^{\infty}$ is monotone and bounded $(0 < S_{2k} < a_1)$. Then it has a limit $\lim\limits_{k\to\infty} S_{2k} = S$, $S \le a_1$. Now consider a partial sum with an odd number: $S_{2k+1} = S_{2k} + a_{2k+1}$. Then, taking into account the ferst condition of the theorem, we have $\lim\limits_{k\to\infty} S_{2k+1} = \lim\limits_{k\to\infty} S_{2k} + \lim\limits_{k\to\infty} a_{2k+1} = S + 0 = S$. Therefore, the series $\sum\limits_{n=1}^{\infty} (-1)^{n-1}\ a_n$ converges and its sum does not exceed the first term a_1.

Remarks. 1. The Leibniz test remains valid even if the inequality $a_n > a_{n+1}$ is satisfied starting from a certain number.

2. Series whose sum can be calculated exactly are not so common. Most often, when solving practical problems, they act as follows: having found out that the series converges, they replace its sum with a n-th partial sum S_n. After all, for sufficiently large values n, the approximate equality $S_n \approx S$ is valid . The error of this approximation is equal to the sum of the remainder of the series r_n. Therefore, the question arises of estimating this remainder. For a series that converges according to the Leibniz test, this question is solved quite simply. After all, the conditions of the theorem 15 are also satisfied for the remainder $(-1)^n a_{n+1} + (-1)^{n+1} a_{n+2} + \ldots = (-1)^n (a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \ldots)$. Then its sum satisfies the inequality $|r_n| \le a_{n+1}$. Thus, the error has the sign of the first discarded term of the series and does not exceed it in absolute value.

3. If condition 1) of theorem 15 is not met, then the series with alternating signs diverges, but if the second condition is not met (the tendency to zero is non-monotonic), then additional research is necessary (the series may converge). For instance, $\frac{1}{2} - 1 + \frac{1}{4} - \frac{1}{3} + \frac{1}{6} - \frac{1}{5} + \frac{1}{8} - \frac{1}{7} + \ldots$ converges and its sum $S = \frac{3}{2}\ln 2 - 1$. Moreover, if a series converges conditionally, then its terms can be rearranged so that it converges to any predetermined number or forced to diverge.

Examples. Study the convergence of series:

1. $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{2^{n+3}(3n+1)}$.

Consider a series of modules $\sum_{n=1}^{\infty}\frac{1}{2^{n+3}(3n+1)}$. Study its convergence by the d'Alembert test. Write down the common term of the series $a_n=\frac{1}{2^{n+3}(3n+1)}$ and the next $a_{n+1}=\frac{1}{2^{n+4}(3n+4)}$. Find $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{2^{n+3}(3n+1)}{2^{n+4}(3n+4)}=\frac{1}{2}<1$. Therefore, the series of modules converges. Then the studied series with alternating signs converges absolutely.

2. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$.

The series of modules $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ diverges $\left(\frac{1}{2}<1\right)$. This means that the series under study with alternating signs is not absolutely convergent. But it can converge conditionally. Let us apply the Leibniz test:

1) $\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$,

2) $1>\frac{1}{\sqrt{2}}>\frac{1}{\sqrt{3}}>\ldots>\frac{1}{\sqrt{n}}>\frac{1}{\sqrt{n+1}}>\ldots$.

So the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges conditionally.

Self-study task

Study the absolutely (conditionally) convergence of the series:

1) $\sum_{n=1}^{\infty}\frac{(-2)^n}{5n+4}$; 2) $\sum_{n=1}^{\infty}(-1)^n\sin\left(\frac{\pi}{n}\right)$; 3) $\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt[3]{8n-1}}{5^n}$; 4) $\sum_{n=1}^{\infty}\frac{(-1)^n\ln n}{n}$.

§5. Functional series

A functional series is a series whose members are functions $u_n(x)$ $\left(n=\overline{1,\infty}\right)$ defined in some domain D. Therefore, a functional series has the form

$$\sum_{n=1}^{\infty}u_n(x)=u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots \qquad (7)$$

At any point $x_0\in D$ the functional series transforms into a numerical series: $u_1(x_0)+u_2(x_0)+\ldots+u_n(x_0)+\ldots$, which may be convergent or divergent. If the numerical series $\sum_{n=1}^{\infty} u_n(x_0)$ converges (diverges), then we say that the functional series $\sum_{n=1}^{\infty} u_n(x)$ converges (diverges) at the point x_0.

Definition. The set of all points at which the series (7) converges is called its region of convergence.

The partial sum of the series (7) $S_n(x)=u_1(x)+u_2(x)+\ldots+u_n(x)$ is a function of the variable x. For any number x belonging to the convergence region, $\lim_{n\to\infty} S_n(x)=S(x)$ holds . The function $S(x)$ is called the sum of the functional series. That is $\sum_{n=1}^{\infty} u_n(x)=u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots=S(x)$.

The remainder of the series $r_n(x)=u_{n+1}(x)+u_{n+2}(x)+\ldots+u_{n+k}(x)+\ldots$ is also a function defined in the convergence region of this series. As we see, $r_n(x)=S(x)-S_n(x)$.

The series (7) is called majored on the segment $[a;b]$, if there exists a convergent numerscal series $\sum_{n=1}^{\infty} a_n$ with positive terms such that $|u_n(x)|\le a_n$ $(n=\overline{1,\infty})$, $x\in[a;b]$. In this case, $\sum_{n=1}^{\infty} a_n$ is called a majorant or a majoring series. For example, the series $\sum_{n=1}^{\infty}\frac{\sin nx}{n^2}$ has a majorant $\sum_{n=1}^{\infty}\frac{1}{n^2}$ since $\frac{|\sin nx|}{n^2}\le\frac{1}{n^2}$ $(n=\overline{1,\infty})$, $x\in(-\infty;+\infty)$ and the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges. It follows directly from the definition that a majored series on the segment $[a;b]$ absolutely converges at all points of this segment.

Let us formulate without proof the main properties of majored series.

Theorem 16. Let the functional series (7) be majored on the segment $[a;b]$. Then for any positive number ε there is a number N such that for all $n\ge N$ the inequality $|r_n(x)|<\varepsilon$ (where $r_n(x)$ is the sum of the n-th remainder of the series (7)) is satisfied immediately for all $x\in[a;b]$.

Theorem 17. If all terms of a major series on segment $[a;b]$are continuous, then its sum $S(x)$ is also continuous on $[a;b]$.

Theorem 18. Let the series (7) of continuous functions be majored on $[a;b]$, $S(x)$ is its sum. Then the series can be integrated term by term over the segment $[a;b]$ and the equality $\int_a^b S(x)dx = \int_a^b u_1(x)dx + \int_a^b u_2(x)dx + \ldots + \int_a^b u_n(x)dx + \ldots$ holds.

Theorem 19. Let the series composed of functions $u_n(x)$, which have continuous derivatives $u_n'(x)$, converges on the segment $[a;b]$. If the series $u_1'(x) + u_2'(x) + \ldots + u_n'(x) + \ldots$ composed of derivatives has majorant on $[a;b]$, then the equality $S'(x) = u_1'(x) + u_2'(x) + \ldots + u_n'(x) + \ldots$ holds.

§6. Power series

A power series is a functional series of the form

$$c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \ldots + c_n(x - x_0)^n + \ldots = \sum_{n=0}^{\infty} c_n(x - x_0)^n,$$

where c_n, x_0 are some numbers. The numbers c_n $\left(n = \overline{1,\infty}\right)$ are called coefficients of a power series. If $x_0 = 0$, then the power series has the form:

$$c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots = \sum_{n=0}^{\infty} c_n x^n.$$

The power series $\sum_{n=0}^{\infty} c_n(x - x_0)^n$ is reduced to the series $\sum_{n=0}^{\infty} c_n t^n$ by substitution $t = x - x_0$, so in the following, when speaking of a power series, we will only mean series of the form $\sum_{n=0}^{\infty} c_n x^n$.

Theorem 20 (Abel theorem).

1. If a power series converges at a point $x = \bar{x}$ $(\bar{x} \neq 0)$, then it converges absolutely at any point x, for which $|x| < |\bar{x}|$.

2. If a power series diverges at a point $x = x$, then it diverges for any x, for which $|x| > |x|$.

Proof. If $\sum_{n=0}^{\infty} c_n \bar{x}^n$ converges, then its general term $c_n \bar{x}^n$ tends to 0 as $n \to \infty$. Then there exists a number M $(M > 0)$ such that $\left|c_n \bar{x}^n\right| \leq M$. Let x is any number for which the following inequality $|x| < |\bar{x}|$ holds, i.e., $\left|\frac{x}{\bar{x}}\right| < 1$. Consider the series $\sum_{n=0}^{\infty} c_n x^n$. For its members we have: $\left|c_n x^n\right| = \left|c_n \bar{x}^n\right| \cdot \left|\frac{x^n}{\bar{x}^n}\right| \leq M \cdot \left|\frac{x}{\bar{x}}\right|^n$. Thus, the moduli of the members of this series do not exceed the members of the convergent geometric series $\sum_{n=0}^{\infty} Mq^n$, $q = \left| \frac{x}{\bar{x}} \right| < 1$. Then the series $\sum_{n=0}^{\infty} \left| c_n x^n \right|$ converges. Therefore, the series $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely.

2. Let $\sum_{n=0}^{\infty} c_n x^n$ diverges at the point x. If it could converge at any point x^*, $\left| x^* \right| > |x|$, then (according to the proof in the first paragraph) it would converge at all points at which $|x| < \left| x^* \right|$. Thus, the series $\sum_{n=0}^{\infty} c_n x^n$ would also converge. The obtained contradiction proves the second part of the theorem.

Let us consider the question of how to find the region of convergence of a power series. First find out at what values x the power series is absolutely convergent. Examine the series of modules $\sum_{n=1}^{\infty} |c_n| |x|^n$ by the d'Alembert test $a_n = |c_n| |x|^n$, $a_{n+1} = |c_{n+1}| |x|^{n+1}$, $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{|c_{n+1}| |x|^{n+1}}{|c_n| |x|^n} = |x| \lim_{n \to \infty} \frac{|c_{n+1}|}{|c_n|}$. The series of modules converges at such values x, that is $|x| \lim_{n \to \infty} \frac{|c_{n+1}|}{|c_n|} < 1$, i.e., at $|x| < \frac{1}{\lim_{n \to \infty} \frac{|c_{n+1}|}{|c_n|}}$. Let us introduce the notation $R = \frac{1}{\lim_{n \to \infty} \frac{|c_{n+1}|}{|c_n|}}$. Therefore, in the interval $|x| < R$ or $x \in (-R; R)$ the power series converges absolutely. If $|x| > R$, then for the series of modules we have $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} > 0$, that is, the series $\sum_{n=1}^{\infty} |c_n| |x|^n$ deverges based on the necessary convergence condition ($\lim_{n \to \infty} a_n = \lim_{n \to \infty} |c_n| |x|^n \neq 0$). Then the general term of the power series also does not tend to $n \to \infty$ at $n \to \infty$. Thus, the power series converges in the interval $(-R; R)$ and diverges at $x \in (-\infty; -R) \cup (R; +\infty)$. The interval $(-R; R)$ is called the convergence interval of the power series, the number R is called the convergence radius. In order to find out whether to add points $x = \pm R$ to the convergence region, they should be put in a series and the resulting numerical series should be examined.

Example. Find the region of convergence of the power series $\sum_{n=1}^{\infty}\frac{3^n x^n}{\sqrt{n+1}}$.

Solving. Let's investigate the series of modules $\sum_{n=1}^{\infty}\frac{3^n |x|^n}{\sqrt{n+1}}$. Let's apply the d'Alembert test:

$$a_n=\sum_{n=1}^{\infty}\frac{3^n |x|^n}{\sqrt{n+1}},\quad a_{n+1}=\sum_{n=1}^{\infty}\frac{3^{n+1}|x|^{n+1}}{\sqrt{n+2}},$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{3^{n+1}|x|^{n+1}}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{3^n|x|^n}= =3|x|\lim_{n\to\infty}\sqrt{\frac{n+1}{n+2}}=3|x|.$$

The interval of convergence of the power series $3|x|<1,\ |x|<\frac{1}{3},\ -\frac{1}{3}<x<\frac{1}{3}$

If $|x|>\frac{1}{3}$, then the series diverges. It remains to study the series $\sum_{n=1}^{\infty}\frac{3^n x^n}{\sqrt{n+1}}$ at $x=\pm\frac{1}{3}$.

At $x=\frac{1}{3}$ we obtain a numerical series $\sum_{n=1}^{\infty}\frac{3^n\cdot\left(\frac{1}{3}\right)^n}{\sqrt{n+1}}=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}}$, which diverges by the comparison test (because it behaves the same as a divergent series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$).

At $x=-\frac{1}{3}$ we obtain a series $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$ with alternating signs. Let us examine it by the Leibniz's test:

1) $\lim_{n\to\infty}\frac{1}{\sqrt{n+1}}=0$,

2) $1>\frac{1}{\sqrt{2}}>\frac{1}{\sqrt{3}}>\ldots>\frac{1}{\sqrt{n+1}}>\frac{1}{\sqrt{n+2}}>\ldots.$

Therefore, the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$ converges. Thus, the region of convergence of the power series is $\left[-\frac{1}{3};\frac{1}{3}\right)$.

Remarks.

1. There are series that converge at $x \in R$ (for example, the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$). If the interval of convergence is $(-\infty;+\infty)$, then we assume that $R = +\infty$.

2. There are no series that diverge at $x \in R$. Any power series converges at least at one point $(x = 0)$. If the region of convergence contains only this one point, then we assume that $R = 0$.

3. If for the series $\sum_{n=1}^{\infty} |c_n||x|^n$ instead d'Alembert tests would have been used the Cauchy root test, then we would obtain a different formula for the radius of convergence: $R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|c_n|}}$.

4. The interval of convergence of the series $\sum_{n=1}^{\infty} c_n (x - x_0)^n$ is given by the inequality $|x - x_0| < R$; $-R < x - x_0 < R$; $x_0 - R < x < x_0 + R$.

Theorem 21. Let the power series $\sum_{n=0}^{\infty} c_n x^n$ converge in the interval $(-R;R)$, then it is majored on any interval $[-r;r]$, where $0 < r < R$.

Proof. In the interval of convergence $(-R;R)$ the power series converges absolutely. That is, for any r $(0 < r < R)$, the numerical series $\sum_{n=1}^{\infty} |c_n| r^n$ converges. But for all $x \in [-r;r]$ we have $|c_n x^n| < |c_n| r^n$ $\left(n = \overline{1,\infty}\right)$. Thus, the power series is majored on the segment $[-r;r]$.

Note that then all properties of majored series can be transferred to the power series.

Theorem 22. The sum of a series $\sum_{n=0}^{\infty} c_n x^n$ is continuous at any point of the convergence interval**.**

Theorem 23. Let a power series $\sum_{n=0}^{\infty} c_n x^n$ have a convergence interval $(-R;R)$ and $S(x)$ is its sum. Then the series obtained by its term-by-term differentiation $\sum_{n=1}^{\infty} n \cdot c_n x^{n-1}$ has the same convergence interval and its sum $\sigma(x) = S'(x)$, $x \in (-R;R)$.

Theorem 24. A power series can be integrated term-by-term over any interval lying inside the convergence interval, that is, for any $x_1, x_2 \in (-R;R): \int\limits_{x_1}^{x_2} \left(\sum\limits_{n=0}^{\infty} c_n x^n \right) dx = \left(\sum\limits_{n=0}^{\infty} c_n \cdot \frac{x^{n+1}}{n+1} \right) \Bigg|_{x_1}^{x_2}$.

Self-study task

Find the region of convergence of the power series

1) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n (x+5)^n}{4n+3}$; 2) $\sum\limits_{n=1}^{\infty} \frac{(x-1)^n}{8^{n+1}}$; 3) $\sum\limits_{n=1}^{\infty} \frac{(4-2x)^n}{n!}$; 4) $\sum\limits_{n=1}^{\infty} \frac{n^2 \ (x-3)^n}{2n-1}$.

§6. Expansion of a function into a power series

Let the function $f(x)$ be the sum of a power series $\sum\limits_{n=0}^{\infty} c_n (x-x_0)^n$. That is, the equality

$$f(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \ldots + c_n(x-x_0)^n + \ldots . \qquad (8)$$

holds in the interval $(x_0 - R;\ x_0 + R)$.

Let us call this equality the expansion of the function $f(x)$ into series by $(x-x_0)$ powers. Let the function $f(x)$ have derivatives of any order at each point of the interval $(x_0 - R; x_0 + R)$. Since the power series can be differentiated term by term at each point of the convergence interval, then, having differentiated both sides of the equality (8), we have:

$$f'(x) = c_1 + 2c_2(x-x_0) + 3c_3(x-x_0)^2 + \ldots + nc_n(x-x_0)^{n-1} + \ldots .$$

We can differentiate both sides of equality (8) any number of times.

$$f''(x) = 2c_2 + 3\cdot 2c_3(x-x_0) + \ldots + n(n-1)c_n(x-x_0)^{n-2} + \ldots;$$

$$f'''(x) = 3\cdot 2\cdot 1\cdot c_3 + 4\cdot 3\cdot 2c_4(x-x_0) + \ldots + n(n-1)(n-2)c_n(x-x_0)^{n-3} + \ldots;$$

…

$$f^{(n)}(x) = n(n-1)\cdot\ldots\cdot 1\cdot c_n + (n+1)n\cdot\ldots\cdot 2c_{n+1}(x-x_0) + \ldots;$$

….

Substituting into these equalities and into equality (8) x_0 instead x, we obtain:

$$f(x_0) = c_0,\ f'(x_0) = 1!\,c_1,\ f''(x_0) = 2!\,c_2, \ldots,\ f^{(n)}(x_0) = n!\,c_n, \ldots$$

So,

$$c_n = \frac{f^{(n)}(x_0)}{n!} \ \left(n = \overline{0,\infty} \right). \qquad (9)$$

Therefore, the following theorem is true.

Theorem 25. If a function $f(x)$ admits expansion into a power series in the interval $(x_0 - R; x_0 + R)$, then this expansion is unique.

That is, the coefficients of the expansion of the function $f(x)$ into a power series by $(x - x_0)$ powers are uniquely found by formulas (9).

The series

$$f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots \quad (10)$$

is called the Taylor series for the function $f(x)$.

If $x_0 = 0$, then the series (10) takes the form:

$$f(0) + \frac{f'(0)}{1!} \cdot x + \frac{f''(0)}{2!} \cdot x^2 + \ldots + \frac{f^{(n)}(0)}{n!} \cdot x^n + \ldots .$$

It is called the Maclaurin series for the function $f(x)$.

As we can see, the Taylor series can be constructed for any function that is differentiable any number of times in some neighborhood of the point x_0. But the equality sign between the function itself and its Taylor series cannot always be set. If $S_n(x)$ is the n-th partial sum of the Taylor series, $r_n(x)$ is the sum of the n-th remainder, then the equality $f(x) = S_n(x) + r_n(x)$ is satisfied only if $\lim_{n\to\infty} r_n(x) = 0$ (since $f(x) = \lim_{n\to\infty} S_n(x)$).

Theorem 26. A Taylor series represents a given function $f(x)$ only if $\lim_{n\to\infty} r_n(x) = 0$.

Expansion of some functions into a Maclaurin series

1. Let's write the Maclaurin series for the function $f(x) = e^x$. Taking into account

$$f(x) = f'(x) = f''(x) = \ldots = f^{(n)}(x) = \ldots = e^x,$$

$$f(0) = f'(0) = f''(0) = \ldots = f^{(n)}(0) = \ldots = e^0 = 1,$$

we have

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$

Let's find the convergence radius of this series:

$$R = \lim_{n\to\infty} \frac{|c_n|}{|c_{n+1}|} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty .$$

Then the region of convergence: $x \in (-\infty; +\infty)$.

Let's estimate the remainder of the series

$$|r_n| \le \frac{|x|^{n+1}}{(n+1)!} + \frac{|x|^{n+2}}{(n+2)!} + \ldots + \frac{|x|^{n+k}}{(n+k)!} + \ldots =$$

$$=\frac{|x|^{n+1}}{(n+1)!}\cdot\left(1+\frac{|x|}{n+2}+\frac{|x|^2}{(n+2)(n+3)}+\ldots+\frac{|x|^{k-1}}{(n+2)(n+3)\cdot\ldots\cdot(n+k)}+\ldots\right)\le$$

$$\le\frac{|x|^{n+1}}{(n+1)!}\left(1+\frac{|x|}{n+2}+\frac{|x|^2}{(n+2)^2}+\ldots+\frac{|x|^{k-1}}{(n+2)^{k-1}}+\ldots\right).$$

Assuming that $0<\frac{|x|}{n+2}<1$ (x since is a finite number and $n\to\infty$) on the right side of the inequality we have a geometric series whose sum is equal to $\frac{1}{1-\frac{|x|}{n+2}}$.

Then $0\le\lim\limits_{n\to\infty}|r_n(x)|\le\lim\limits_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!}\cdot\frac{1}{1-\frac{|x|}{n+2}}=0$. Thus, $r_n(x)\to 0$ at $n\to\infty$.

Therefore, at $x\in(-\infty;+\infty)$ the expansion

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots \tag{11}$$

is correct.

2. Similarly, we can obtain the expansion

$$\sin x=\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\ldots, x\in(-\infty;+\infty). \tag{12}$$

3. Differentiating both sides of equality (12), we obtain the equality:

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots+\frac{(-1)^n x^{2n}}{(2n)!}+\ldots, x\in(-\infty;+\infty). \tag{13}$$

4. The expansion of the function $f(x)=(1+x)^m$ (where m is constant) is called binomial. Let's calculate its coefficients:

$f(x)=(1+x)^m,\quad f(0)=1,$

$f'(x)=m(1+x)^{m-1},\quad f'(0)=m,$

$f''(x)=m(m-1)(1+x)^{m-2},\quad f''(0)=m(m-1),\ \ldots,$

$f^{(n)}(x)=m(m-1)\ldots(m-n+1)(1+x)^{m-n},\quad f^{(n)}(0)=m(m-1)\ldots(m-n+1)\ \ldots.$

So the binomial series has the form:

$$1+\frac{m}{1!}\cdot x+\frac{m(m-1)}{2!}\cdot x^2+\ldots+\frac{m(m-1)\ldots(m-n+1)}{n!}\cdot x^n+\ldots.$$

Let's calculate its radius of convergence

$$R=\lim_{n\to\infty}\left|\frac{m(m-1)\ldots(m-n+1)}{n!}\cdot\frac{(n+1)!}{m(m-1)\ldots(m-n+1)(m-n)}\right|=\lim_{n\to\infty}\left|\frac{n+1}{m-n}\right|=1.$$

Therefore, the binomial series has convergence interval $(-1;1)$. It can be proved that $\lim\limits_{n\to\infty}\left|r_n(x)\right|=0$ for $x\in(-1;1)$. Then we obtain the equality

$$(1+x)^m=1+\frac{m}{1!}\cdot x+\frac{m(m-1)}{2!}\cdot x^2+\ldots+\frac{m(m-1)\ldots(m-n+1)}{n!}\cdot x^n+\ldots,\qquad (14)$$

$x\in(-1;1)$.

The behavior of the series on the ends of the interval depends on index m (for $m\geq 0$ convergance region is $x\in[-1;1]$, for $-1<m<0$ convergance region is $x\in(-1;1]$, for $m\leq -1$ convergance region is $x\in(-1;1)$).

Note that if m is a natural number, the binomial series becomes a polynomial (since the coefficients at all higher m powers are zero), and formula (14) turns into the Newton binomial formula

$$(1+x)^m=1+\frac{m}{1!}\cdot x+\frac{m(m-1)}{2!}\cdot x^2+\ldots+\frac{m(m-1)...2}{(m-1)!}\cdot x^{m-1}+x^m .$$

5. Let's write the expansion of the function $\frac{1}{1+x}=(1+x)^{-1}$ according to the formula (14):

$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots+(-1)^n x^n+\ldots .\qquad (15)$$

Let us integrate both sides of this equality within the range from 0 to x. We obtain

$$\ln(1+x)=\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+\frac{(-1)^n x^{n+1}}{n+1}+\ldots .\qquad (16)$$

Region of convergence is $(-1;1]$.

6. If in formula (15) we replace x with x^2 and also integrate both parts, we obtain the expansion of the function $f(x)=arctg\ x$.

$$arctg\ x=\frac{x}{1}-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots+\frac{(-1)^n x^{2n+1}}{2n+1}+\ldots .\qquad (17)$$

Convergence region is $[-1;1]$.

Let's consider some examples of applying power series to approximate calculations.

Example. Calculate the approximate value of a function $f(x)=e^x$ at point $x=-\frac{1}{4}$ with accuracy $\varepsilon=0,01$.

Solving. Let us use the expansion (11) for $x=-\frac{1}{4}$. We obtain

$$e^{-\frac{1}{4}}=1-\frac{1/4}{1!}+\frac{1/16}{2!}-\frac{1/64}{3!}+\ldots=1-\frac{1}{4}+\frac{1}{32}-\frac{1}{384}+\ldots.$$

Note that $|a_3|=\frac{1}{32}>\varepsilon$ and $|a_4|=\frac{1}{384}<\varepsilon$. Replace the sum of the series by the sum of its first three terms $e^{-\frac{1}{4}}\approx 1-\frac{1}{4}+\frac{1}{32}=\frac{25}{32}$. This numerical series satisfies the conditions of the Leibniz test, therefore the approximation error does not exceed the modulus of the first discarded term of the series $\left(\Delta\le\frac{1}{384}<0,01\right)$. Therefore, the required accuracy is ensured.

Example. Calculate the approximate definite integral $\int_0^{0,5}\frac{\sin\left(2x^3\right)}{x}dx$ by expanding the integrand into x power series and integrating its first two terms. Estimate the error.

Solving. Note that the antiderivative of the function $\frac{\sin\left(2x^3\right)}{x}$ is not expressed in terms of elementary functions, i.e., this integral cannot be calculated by the classic methods. Using formula (12), we have:

$$\int_0^{0,5}\frac{\sin\left(2x^3\right)}{x^2}dx=\int_0^{0,5}\frac{1}{x^2}\left(\frac{2x^3}{1!}-\frac{8x^9}{3!}+\frac{32x^{15}}{5!}-\ldots\right)dx=\int_0^{0,5}\left(2x-\frac{4x^7}{3}+\frac{4x^{13}}{15}-\ldots\right)dx=$$

$$=\left(x^2-\frac{x^8}{6}+\frac{2x^{14}}{105}-...\right)\Bigg|_0^{0,5}=\frac{1}{4}-\frac{1}{1536}+\frac{1}{860160}-\ldots\approx\frac{1}{4}-\frac{1}{1536}=\frac{383}{1536}.$$

Let us estimate the approximation error: $\Delta\le|a_3|=\frac{1}{860160}$.

Example. Write the first three non-zero terms of the expansion in a power series of the Cauchy problem solution: $y'=y^2-3\sin x-4,\ y(0)=-2$.

Solving.. As we can see, this differential equation cannot be attributed to any of the types considered in the section *Defferential Equations*, so we have no way of finding its general solution. We will look for a solution $y(x)$ to the Cauchy problem in the form of a Maclaurin series:

$$y(x)=y(0)+\frac{y'(0)}{1!}\cdot x+\frac{y''(0)}{2!}\cdot x^2+\ldots+\frac{y^{(n)}(0)}{n!}\cdot x^n+\ldots.$$

The first two terms are given by the initial condition $y(0)=-2$ and the differential equation

$$y' = y^2 - 3\sin x - 4, \quad y'(0) = (-2)^2 - 3\sin 0 - 4 = 0.$$

Differentiate both parts of the differential equation. We obtain

$$y'' = 2yy' - 3\cos x, \quad y''(0) = -3;$$

$$y''' = 2(y')^2 + 2yy'' - 3\sin x, \quad y'''(0) = 12.$$

Then the Cauchy problem solution has the form

$$y = -2 + \frac{-3}{2!}\cdot x^2 + \frac{12}{3!}\cdot x^3 + \ldots \text{ or } y = -2 - \frac{3}{2}x^2 + 2x^3 + \ldots.$$

Self-study task

1) Calculate the approximate value with accuracy $\varepsilon = 0{,}001$

a) $arctg\ (0.4)$; b) $\sqrt[5]{1.2}$; c) $\cos 7^\circ$; d) $\int_0^{0,5} \sin(x^2)\,dx$; e) $\int_0^1 \frac{1-\cos(x^3)}{x}\,dx$.

2) Write the first three non-zero terms of the expansion in a power series of the Cauchy problem solution:

a) $y'' + xy' = 1 - y,\ y(0) = 1,\ y'(0) = -1;$

b) $y' = 2\cos x - xy^2,\ y(0) = 1.$

Reference

1. Gavdzinski V.N., Korobova L.N. Higher mathematics. Hart I. Calculus with Linear Algebra and Analytic Geometry: Textbook. Odessa. 2016. 326 p.